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Reflector design for two-variable beam shaping in the hyperbolic case

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Abstract. The design of a reflector capable of producing a generalized two-variable beam shape when illuminated by a point source is investigated under the geometric-optics approximation. The problem is formulated as a mapping problem between points on a unit sphere in which areas are related by energy considerations. The solution of a resulting set of nonlinear partial differential equations is required. An investigation of the hyperbolic form of these equations shows that they can be reduced to a set of quasi-linear first-order partial differential equations which can be solved subject to initial conditions.

1. Introduction

A new method has been introduced recently by Norris and Westcott (1974) for the design of a reflector surface under the geometric-optics approximation, to produce a beam shape which varies as a prescribed function of two variables when the reflector is illuminated by an isotropic point source.

The theoretical foundations of the method are given in Westcott and Norris (1975) who consider the following problem. If $D(u^1, u^2)$ is the ratio between the reflector far-field and incident power densities G, I respectively, where u^1, u^2 are generalized coordinates which parametrize the reflected ray direction, can an explicit surface $r(u^1, u^2)$ be found such that the laws of energy conservation and geometrical reflection are satisfied?

Some simplifying assumptions are made in the above reference, among which I is assumed constant corresponding to an isotropic point source, and it is shown that the problem may be reduced to solving a nonlinear second-order partial differential equation of the Monge–Ampère type

$$\sigma_{\theta\theta}\sigma_{\phi\phi} - \sigma_{\theta\phi}^2 = a\sigma_{\theta\theta} + 2b\sigma_{\theta\phi} + c\sigma_{\phi\phi} + d \pm eD \tag{1}$$

when spherical polars $(u^1, u^2) \equiv (\theta, \phi)$ are used to define reflected ray directions. In this equation a, b, c, d and e are *explicit* functions of θ, σ_θ, σ_ϕ, and both e and $D(\theta, \phi)$ are positive functions. A formula for the reflector surface $\boldsymbol{r}(\theta, \phi, \sigma, \sigma_\theta, \sigma_\phi)$ is developed and computed from solutions of (1).

The choice of the positive or negative sign affixed to the last term of (1) makes the equation elliptic or hyperbolic respectively, and previous work has examined the elliptic form of the equation and has introduced closed boundary conditions to enable numerical solutions to be obtained.

In this paper we describe an alternative but equivalent treatment which has advantages when hyperbolic forms of the equation are considered. The approach results ultimately in a formulation in terms of an initial-value problem involving a set of four, simultaneous quasi-linear first-order partial differential equations.

The main benefits of this treatment are that questions of existence and uniqueness of solutions can be handled by classical theorems, and the computation of reflector surfaces may be achieved using standard methods of analysis.

In § 2 the problem is formulated by considering a 1 : 1 mapping between points on a unit sphere. Energy considerations imply that areas on the sphere are increased by a factor D under the transformation. A further equation is derived from the geometrical law of reflection. Some exact solutions of these equations are derived in § 3. It is not proposed that these solutions are necessarily significant from the point of view of physical applications, but they do provide a useful check on numerical methods.

The development of the equivalent set of quasi-linear first-order partial differential equations for the hyperbolic case is given in § 4, where a method related to the method of characteristics is used. The equations are solved subject to the dependent variables being given initial values. This procedure is discussed in § 5.

Some general results about reflector curvature are given in § 6, and finally some conclusions are presented in § 7.

2. Formulation of mapping problem

In figure 1 a sphere of unit radius and centre O is drawn, where O is the point source of incident rays. The points P, Q are the end points of unit vectors z, y drawn from O.

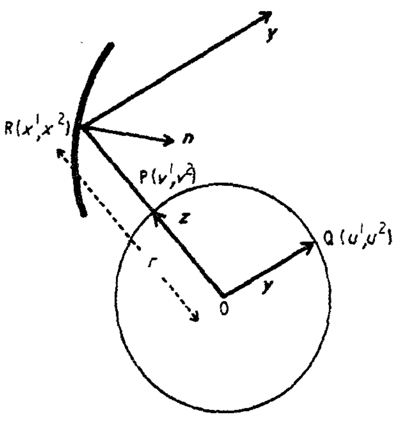

Figure 1. Diagram showing incident and reflected ray directions.

The unit vector z is in the direction of the incident ray, $x = rz$ is the position vector of the point of reflection R, and y is in the direction of the reflected ray. Our aim is to find x as a function of y so that a given far-field power density pattern $G(y)$ is obtained from a given source power pattern $I(z)$. Geometrically this means that the mapping $\chi : Q \to P$ has to alter areas by a factor

$$D = G/I.$$

To express this condition as a differential equation we choose a system of coordinates for P and a (possibly different) system for Q. Let (v^1, v^2) denote the coordinates of P and (u^1, u^2) those of Q with respect to the corresponding coordinate systems. Thus z, I can be expressed as functions of (v^1, v^2) and y, G as functions of (u^1, u^2). We write

$$z_i = \partial z/\partial v^i, \qquad y_i = \partial y/\partial u^i,$$
$$h_{ij} = z_i \cdot z_j, \qquad g_{ij} = y_i \cdot y_j, \qquad i, j = 1, 2$$

and denote the determinants of the matrices h_{ij}, g_{ij} by h, g respectively. We shall assume that the coordinate systems have been chosen so that the triple scalar products $[z, z_1, z_2]$, $[y, y_1, y_2]$ are positive. It is easy to see that they are then equal to $\sqrt{h}$ and $\sqrt{g}$ respectively.

With this notation the area condition is that the mapping χ, when expressed in terms of coordinates as $(u^1, u^2) \to (v^1, v^2)$, should satisfy one of the differential equations

$$\frac{\partial v^1}{\partial u^1}\frac{\partial v^2}{\partial u^2} - \frac{\partial v^1}{\partial u^2}\frac{\partial v^2}{\partial u^1} = \pm\left(\frac{g}{h}\right)^{1/2} D. \tag{2}$$

The differential equation (2) is not the only restriction on χ because the mapping must also be realized by reflection in a reflector. To express this condition as a differential equation we first introduce some notation. The symbols $\mathrm{d}/\mathrm{d}u^i$, $i = 1, 2$, will denote the partial derivatives of a function whose variables have been expressed in terms of u^1 and u^2 only. Thus, for example

$$\mathrm{d}z/\mathrm{d}u^i = \sum_j z_j(\partial v^j/\partial u^i), \qquad j = 1, 2.$$

It is clear that the vector $x - ry$ is a normal to the reflector at R. Since the vector $\mathrm{d}x/\mathrm{d}u^i$ is tangential to the reflector it follows that

$$(\mathrm{d}x/\mathrm{d}u^i) \cdot (x - ry) = 0, \qquad i = 1, 2.$$

These conditions can easily be modified to give

$$\mathrm{d}\sigma/\mathrm{d}u^i = y \cdot (\mathrm{d}z/\mathrm{d}u^i)/\Lambda \tag{3}$$

where $\sigma = \log r$ and $\Lambda = 1 - y \cdot z$. It is convenient to introduce the function

$$\Psi = -\log \Lambda \tag{4}$$

which is a function of all four variables v^1, v^2, u^1, u^2. Because $\partial\Psi/\partial v^i = y \cdot z_i/\Lambda$ it follows that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}u^i} = \sum_j \frac{\partial\Psi}{\partial v^j}\frac{\partial v^j}{\partial u^i}, \qquad i, j = 1, 2. \tag{5}$$

The mapping χ satisfies the differential equation obtained from the integrability conditions

$$\mathrm{d}^2\sigma/\mathrm{d}u^1\,\mathrm{d}u^2 = \mathrm{d}^2\sigma/\mathrm{d}u^2\,\mathrm{d}u^1 \tag{6}$$

and a straightforward calculation using (4) shows that this is

$$\Psi_{12}\frac{\partial v^1}{\partial u^1} - \Psi_{11}\frac{\partial v^1}{\partial u^2} + \Psi_{22}\frac{\partial v^2}{\partial u^1} - \Psi_{21}\frac{\partial v^2}{\partial u^2} = 0 \tag{7}$$

where we have written Ψ_{ij} for the mixed derivative $\partial^2\Psi/\partial v^i \partial u^j$.

We can therefore attack our problem in two stages. We first solve the partial differential equations (2) and (7) to obtain v^1, v^2 as functions of u^1, u^2, and then find σ (and hence r) by integrating (5). In order to apply numerical methods to the solution of (2) and (7) we have to impose initial- or boundary-value conditions to force uniqueness. The choice of such conditions will depend on the nature of the differential equations. We now show that they are of *hyperbolic* or *elliptic* type, according to the choice of + or − sign in (2).

The characteristics of the equations (for a given solution) are the integral curves of the differential equations $\mathrm{d}u^2/\mathrm{d}u^1 = \kappa$, where κ is a root of the quadratic equation

$$\begin{vmatrix} -\Psi_{11}-\kappa\Psi_{12} & -\Psi_{21}-\kappa\Psi_{22} \\ \dfrac{\partial v^2}{\partial u^1}-\kappa\dfrac{\partial v^2}{\partial u^2} & \dfrac{\partial v^1}{\partial u^1}+\kappa\dfrac{\partial v^1}{\partial u^2} \end{vmatrix} = 0. \tag{8}$$

The differential equations are of hyperbolic, parabolic or elliptic type according to whether these roots are real and distinct, coincident or complex. A calculation, using the equations (2) and (7), shows that the discriminant of the quadratic is

$$\mp|\Psi_{ij}|D(g/h)^{1/2}$$

depending on the sign in (2). It is shown in the appendix that the determinant $|\Psi_{ij}|$ is equal to $-(gh)^{1/2}/\Lambda^2$ and is therefore negative. Our statement is a consequence of this fact.

We do not make a detailed study of the elliptic case in this paper. The hyperbolic case will be considered in §§ 4 and 5, where we shall explain a method of finding solutions subject to given initial values.

3. Some explicit solutions

We choose a system of spherical polar coordinates on the unit sphere corresponding to the axes OX, OY, OZ shown in figure 2(a). Let (α, β), (θ, ϕ) denote, respectively, the coordinates of the points P, Q in this system. In the notation of § 2

$$v^1 = \alpha, \qquad v^2 = \beta, \qquad u^1 = \theta, \qquad u^2 = \phi.$$

We find

$$z = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$$

$$y = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$h = \sin^2\alpha, \qquad g = \sin^2\theta$$

$$\Lambda = 1 - \sin\alpha\sin\theta\cos(\beta-\phi) - \cos\alpha\cos\theta$$

$$\Lambda^2\Psi_{11} = (\cos\alpha\cos\theta - 1)\cos(\beta-\phi) + \sin\alpha\sin\theta$$

$$\Lambda^2\Psi_{12} = \sin\theta(\cos\alpha - \cos\theta)\sin(\beta-\phi)$$

$$\Lambda^2\Psi_{21} = \sin\alpha(\cos\alpha - \cos\theta)\sin(\beta-\phi)$$

$$\Lambda^2\Psi_{22} = \sin\alpha\sin\theta[(1-\cos\theta\cos\alpha)\cos(\beta-\phi) - \sin\alpha\sin\theta].$$

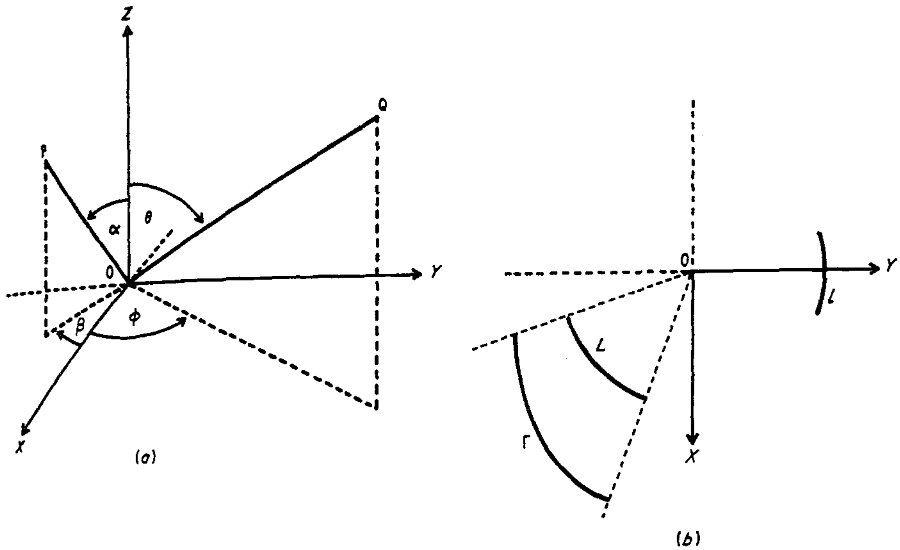

Figure 2. (*a*) Diagram showing coordinate system. (*b*) The initial lines *l*, *L*, Γ on the plane $Z=0$.

We shall construct solutions of equations (2) and (7), for particular functions D, by the following method. We try to find solutions of equation (7) in some special form. If such solutions exist we can use equation (2) to *define* a corresponding function D.

Equations (7) and (2) become, in spherical polars,

$$A\alpha_\theta + B\alpha_\phi + C\beta_\theta + E\beta_\phi = 0 \tag{9}$$

$$\alpha_\theta\beta_\phi - \alpha_\phi\beta_\theta = \pm D \sin\theta/\sin\alpha \tag{10}$$

where we have put

$$A=\Lambda^2\Psi_{12}, \qquad B=-\Lambda^2\Psi_{11}, \qquad C=\Lambda^2\Psi_{22}, \qquad E=-\Lambda^2\Psi_{21}.$$

After a solution $\alpha(\theta, \phi)$, $\beta(\theta, \phi)$ is obtained from (9) and (10) the reflector surface $r(\theta, \phi)$ can be derived by first integrating equations (5) which may be rewritten

$$\sigma_\theta = -(\tilde{X}\alpha_\theta + \tilde{Y}\beta_\theta)/\Lambda \tag{11}$$

$$\sigma_\phi = -(\tilde{X}\alpha_\phi + \tilde{Y}\beta_\phi)/\Lambda \tag{12}$$

where

$$\tilde{X} = \sin\alpha\cos\theta - \sin\theta\cos\alpha\cos(\beta-\phi)$$

$$\tilde{Y} = \sin\alpha\sin\theta\sin(\beta-\phi)$$

and then putting

$$r(\theta, \phi) = \exp\sigma(\theta, \phi).$$

We now consider some special cases.

Case (i). $\alpha=\theta$, $\beta(\phi)$ *arbitrary*

For this case $A=E=0$ and $\alpha_\phi=\beta_\theta=0$ so that (9) is satisfied identically and (10) yields $D=|\beta_\phi|$, indicating a θ-independent arbitrary $D(\phi)$. The differential equations are of elliptic or hyperbolic type according to whether $\beta_\phi<0$ or >0. The corresponding reflector surface is obtained by integrating (11) and (12) to yield

$$\sigma=-\log\sin\theta+\int_{\pi/2}^{\phi}\cot\tfrac{1}{2}(\phi-\beta)(\mathrm{d}\beta/\mathrm{d}\phi)\,\mathrm{d}\phi \tag{13}$$

where we have imposed the condition $\sigma=0$ and hence $r=1$ when $\theta=\phi=\pi/2$.

We examine in particular the transformation $\alpha=\theta$, $\beta=k(\phi-\pi/2)-\pi/2$ where k is a non-zero constant. The transformation is arranged so that it takes the point $\theta=\pi/2$, $\phi=\pi/2$ to the point $\alpha=\pi/2$, $\beta=-\pi/2$. Then $D=|k|$ and, for $k\neq1$,

$$r=\exp\sigma=(\sin\theta)^{-1}\{\sin\tfrac{1}{4}[2(1-k)\phi+(1+k)\pi]\}^{2k/(1-k)}$$

or, in terms of α, β,

$$r=(\sin\alpha)^{-1}\{\sin\tfrac{1}{4}k^{-1}[2(1-k)\beta+(1+k)\pi]\}^{2k/(1-k)}.$$

Case (ii). $\operatorname{Sin}(\beta-\phi)=0$, $\alpha(\theta)$ *arbitrary*

We take $\beta=\phi-\pi$ and find that $A=E=0$, and since $\alpha_\phi=\beta_\theta=0$, equation (9) is satisfied identically. Then

$$D=(\sin\alpha/\sin\theta)|\alpha_\theta|$$

which is a ϕ-independent arbitrary function of θ. The differential equations are of elliptic or hyperbolic type according to whether $\alpha_\theta<0$ or >0. Equations (11) and (12) become

$$\sigma_\theta=-\alpha_\theta\cot\tfrac{1}{2}(\alpha+\theta),\qquad \sigma_\phi=0$$

whence

$$r=\exp\left(\int_{\theta}^{\pi/2}\cot\tfrac{1}{2}(\alpha+\theta)(\mathrm{d}\alpha/\mathrm{d}\theta)\,\mathrm{d}\theta\right)$$

taking $r=1$ when $\theta=\pi/2$. If r is expressed in terms of α and β it will depend on α only. Consequently the reflector is axially symmetric about the OZ axis.

We examine in particular the transformation

$$\alpha=k(\theta-\pi/2)+\pi/2,\qquad \beta=\phi-\pi$$

where k is a nonzero constant. Then

$$D=\frac{|k|\sin[k(\theta-\pi/2)+\pi/2]}{\sin\theta}$$

and, for $k\neq-1$,

$$r=\{\sin\tfrac{1}{4}k^{-1}[2(k+1)\alpha+(k-1)\pi]\}^{-2k/(1+k)}.$$

Case (iii). $\alpha = \alpha(\theta)$, $\beta = \beta(\phi)$ (*but excluding* (*i*), (*ii*))

Equations (9) and (10) become

$$\sin\theta \frac{d\alpha}{d\theta} - \sin\alpha \frac{d\beta}{d\phi} = 0$$

$$\frac{d\alpha}{d\theta}\frac{d\beta}{d\phi} = \pm D \frac{\sin\theta}{\sin\alpha}.$$

To satisfy the first equation we separate variables and put

$$\frac{d\beta}{d\phi} = k, \qquad \frac{\sin\theta}{\sin\alpha}\frac{d\alpha}{d\theta} = k$$

where k is an arbitrary constant. It follows from the second equation that the system is of hyperbolic type with $D = k^2 \sin^2\alpha/\sin^2\theta$. The differential equations for α, β can easily be integrated and, if we impose the initial conditions $\alpha(\pi/2) = \pi/2 = -\beta(\pi/2)$, we obtain

$$\alpha = 2\tan^{-1}(u^k), \qquad \beta = k(\phi - \pi/2) - \pi/2$$

where $u = \tan\frac{1}{2}\theta$. The expression for D in terms of u is

$$D = k^2[u^{k-1}(1+u^2)/(1+u^{2k})]^2.$$

For $k > 1$, D has a maximum at $\theta = \pi/2$ ($u = 1$) and monotonically decreases to zero as $\theta \to 0, \pi$.

Equations (11) and (12) can be integrated to yield an explicit form for the reflector surface. If, for simplicity, we take $k = 2$ and integrate with $r = 1$ when $\theta = \pi/2$, $\phi = \pi/2$ we obtain

$$r = 8(1+u^4)/(1+u^2+2u\sin\phi)^2.$$

Case (iv). $\alpha = \alpha(\phi)$, $\beta = \beta(\theta)$

Equations (9) and (10) become

$$\frac{d\alpha}{d\phi} + \sin\alpha \sin\theta \frac{d\beta}{d\theta} = 0$$

$$-\frac{d\alpha}{d\phi}\frac{d\beta}{d\theta} = \pm D \frac{\sin\theta}{\sin\alpha}.$$

Separating variables to satisfy the first equation we put

$$\frac{d\alpha}{d\phi} = k\sin\alpha, \qquad \sin\theta \frac{d\beta}{d\theta} = -k$$

where k is an arbitrary constant. It follows from the second equation that the system is of hyperbolic type with $D = k^2 \sin^2\alpha/\sin^2\theta$. The differential equations for α, β can easily be integrated and, if we impose the initial conditions

$$\alpha(0) = \pi/2, \qquad \beta(\pi/2) = 0,$$

we obtain

$$\alpha = 2\tan^{-1}(\exp k\phi), \qquad \beta = -k\log(\tan\tfrac{1}{2}\theta).$$

The corresponding expression for D in terms of θ, ϕ is

$$D = k^2 \operatorname{sech}^2 k\phi \operatorname{cosec}^2\theta.$$

Therefore D has a sech^2 profile with θ constant, and a cosec^2 profile with ϕ constant.

An explicit form for the reflector surface is not forthcoming in this case, but equations (11) and (12) may be integrated numerically. Indeed this has been done in order to check some more general computer programs to be discussed in a subsequent paper.

4. Reduction to a quasi-linear system

In order to solve the mapping problem formulated in § 2 we have to solve the simultaneous differential equations (2) and (7). Our main aim in this paper is to show how standard numerical methods can be applied to give solutions in the hyperbolic case. An essential step is the replacement of the system (2) and (7) by a set of partial differential equations which are linear in the derivatives. We explain this step in the present section.

In the hyperbolic case the quadratic (8) yields distinct real roots κ_1, κ_2 and by integrating the two differential equations

$$\mathrm{d}u^2/\mathrm{d}u^1 = \kappa_1, \qquad \mathrm{d}u^2/\mathrm{d}u^1 = \kappa_2$$

we obtain two families of characteristics $\xi(u^1, u^2) = \text{constant}$, $\zeta(u^1, u^2) = \text{constant}$. We now regard the functions ξ, ζ as independent variables. It can be shown that u^1, u^2, v^1, v^2, considered as functions of ξ, ζ, satisfy a quasi-linear system which we write in matrix form as

$$\begin{bmatrix} A & C \\ B & E \end{bmatrix} \begin{bmatrix} v^1_\xi & v^1_\zeta \\ v^2_\xi & v^2_\zeta \end{bmatrix} = \Delta \begin{bmatrix} u^1_\xi & -u^1_\zeta \\ u^2_\xi & -u^2_\zeta \end{bmatrix} \tag{14}$$

where

$$A = \Lambda^2\Psi_{12}, \qquad B = -\Lambda^2\Psi_{11}, \qquad C = \Lambda^2\Psi_{22}, \qquad E = -\Lambda^2\Psi_{21}.$$

and Δ is a positive function defined by

$$\Delta^2 = (BC - AE)D(g/h)^{1/2}.$$

It will be convenient when we consider initial conditions in § 5 to have equation (14) expressed in terms of variables x, y where

$$\xi = x - y, \qquad \zeta = x + y.$$

It is easy to check that this expression is

$$\begin{bmatrix} A & C \\ B & E \end{bmatrix} \begin{bmatrix} v^1_x & v^1_y \\ v^2_x & v^2_y \end{bmatrix} = -\Delta \begin{bmatrix} u^1_y & u^1_x \\ u^2_y & u^2_x \end{bmatrix}. \tag{15}$$

We have not given the detailed derivation of (14) and (15) because we use only the converse result. That is, we use the fact that any solution of (15) such that the Jacobian determinant

$$J = \begin{vmatrix} u^1_x & u^1_y \\ u^2_x & u^2_y \end{vmatrix}$$

nonzero leads to a solution of the system (2) and (7). We now justify this statement in detail.

Suppose we are given a solution

$$v^1 = v^1(x, y), \qquad v^2 = v^2(x, y), \qquad u^1 = u^1(x, y), \qquad u^2 = u^2(x, y)$$

of equation (15). The condition $J \neq 0$ implies that the last two equations can be solved to obtain x and y as functions of u^1, u^2. By substituting for x and y in the first two equations we obtain v^1 and v^2 as functions of u^1, u^2. We shall show that these functions satisfy the equations (2) (with the + sign) and (7).

We multiply equation (15) by the matrix of partial derivatives

$$\begin{bmatrix} \partial x/\partial u^1 & \partial x/\partial u^2 \\ \partial y/\partial u^1 & \partial y/\partial u^2 \end{bmatrix}$$

to obtain

$$\begin{bmatrix} A & C \\ B & E \end{bmatrix}\begin{bmatrix} \partial v^1/\partial u^1 & \partial v^1/\partial u^2 \\ \partial v^2/\partial u^1 & \partial v^2/\partial u^2 \end{bmatrix} = -\Delta\begin{bmatrix} u_y^1 & u_x^1 \\ u_y^2 & u_x^2 \end{bmatrix}\begin{bmatrix} \partial x/\partial u^1 & \partial x/\partial u^2 \\ \partial y/\partial u^1 & \partial y/\partial u^2 \end{bmatrix}. \tag{16}$$

Because

$$\begin{bmatrix} \partial x/\partial u^1 & \partial x/\partial u^2 \\ \partial y/\partial u^1 & \partial y/\partial u^2 \end{bmatrix} = \begin{bmatrix} u_x^1 & u_y^1 \\ u_x^2 & u_y^2 \end{bmatrix}^{-1} = \frac{1}{J}\begin{bmatrix} u_y^2 & -u_y^1 \\ -u_x^2 & u_x^1 \end{bmatrix}$$

equation (16) can be written as

$$\begin{bmatrix} A & C \\ B & E \end{bmatrix}\begin{bmatrix} \partial v^1/\partial u^1 & \partial v^1/\partial u^2 \\ \partial v^2/\partial u^1 & \partial v^2/\partial u^2 \end{bmatrix} = -\frac{\Delta}{J}\begin{bmatrix} u_y^1 & u_x^1 \\ u_y^2 & u_x^2 \end{bmatrix}\begin{bmatrix} u_y^2 & -u_y^1 \\ -u_x^2 & u_x^1 \end{bmatrix}.$$

By taking the determinants of each side in the above equation we obtain

$$(AE - BC)\left(\frac{\partial v^1}{\partial u^1}\frac{\partial v^2}{\partial u^2} - \frac{\partial v^1}{\partial u^2}\frac{\partial v^2}{\partial u^1}\right) = -\Delta^2$$

and equation (2) follows from the definition of Δ^2.

By equating the diagonal terms of each side in the above equation we obtain

$$A\frac{\partial v^1}{\partial u^1} + C\frac{\partial v^2}{\partial u^1} = -\frac{\Delta}{J}(u_y^1 u_y^2 - u_x^1 u_x^2)$$

$$B\frac{\partial v^1}{\partial u^2} + E\frac{\partial v^2}{\partial u^2} = -\frac{\Delta}{J}(-u_y^2 u_y^1 + u_x^2 u_x^1)$$

and equation (7) follows by addition.

5. Initial conditions

In order to compute solutions of the system (2) and (7) in the hyperbolic case we force uniqueness by initial conditions. That is, we require that a given curve $u^1 = \phi^1(t)$, $u^2 = \phi^2(t)$ should map under the transformation χ of § 2 into a given curve $v^1 = \psi^1(t)$,

$v^2 = \psi^2(t)$. We explain how to obtain such a solution from the quasi-linear set (15), ie

$$\begin{aligned} A(\partial v^1/\partial x) + C(\partial v^2/\partial x) &= -\Delta(\partial u^1/\partial y) \\ B(\partial v^1/\partial x) + E(\partial v^2/\partial x) &= -\Delta(\partial u^2/\partial y) \\ A(\partial v^1/\partial y) + C(\partial v^2/\partial y) &= -\Delta(\partial u^1/\partial x) \\ B(\partial v^1/\partial y) + E(\partial v^2/\partial y) &= -\Delta(\partial u^2/\partial x). \end{aligned} \tag{15}$$

The characteristics of the system (15) are $y = \pm x + \text{constant}$ so that $y = 0$ is not a characteristic. Consequently, standard theory (see, for example, Courant and Hilbert 1962) implies that there are unique solutions of (15) such that

$$u^i(x, 0) = \phi^i(x), \qquad v^i(x, 0) = \psi^i(x), \qquad i = 1, 2.$$

The work in § 4 shows that, provided the Jacobian J is not zero on the curve $y = 0$, we can now construct the required solution to the system (2) and (7). By using equations (15), the condition on J is easily shown to be the same as

$$\left(A\frac{\mathrm{d}\psi^1}{\mathrm{d}x} + C\frac{\mathrm{d}\psi^2}{\mathrm{d}x}\right)\frac{\mathrm{d}\phi^2}{\mathrm{d}x} - \left(B\frac{\mathrm{d}\psi^1}{\mathrm{d}x} + E\frac{\mathrm{d}\psi^2}{\mathrm{d}x}\right)\frac{\mathrm{d}\phi^1}{\mathrm{d}x} \neq 0 \tag{17}$$

on the curve $y = 0$.

Given the initial conditions, the solution of the system (15) can be computed by a Lax–Wendroff method (Mitchell 1969), and we shall present some numerical results in a subsequent paper. It is clear that the design of a reflector (for a given function D) is dependent on the choice of the initial conditions. We conclude this section by describing the choice we have made in some of our numerical work.

We shall use the spherical polar coordinate system introduced in § 3. We choose the initial curve l defined by $\theta = \pi/2$, $\phi = t$ and we require that this curve should map under χ into a curve L defined by $\alpha = \pi/2$, $\beta = f(t)$. The physical meaning of this choice is that incident rays in the $Z = 0$ plane are reflected within the same plane.

The function $f(t)$ is still at our disposal and we will explain later how we have determined it. The condition (17) imposes some restraints, for, from the formulae given at the beginning of § 3, we find that on the curve $y = 0$

$$A = E = 0, \qquad B = C = \cos(f(x) - x) - 1$$

and consequently (17) becomes

$$[1 - \cos(f(x) - x)]f'(x) \neq 0. \tag{18}$$

Once the system (15) has been integrated, equations (11) and (12) can be used to obtain the partial derivatives $\partial\sigma/\partial x$, $\partial\sigma/\partial y$ of σ regarded as a function of x, y. Consequently σ is determined to within an additive constant and this implies that r is determined to within a multiplicative constant. The numerical determination of σ presents no difficulties.

We note here that $\sigma(x, 0) = \tau(x)$ is determined by the initial conditions. To show this we regard σ as a function of α, β. It follows from (11) and (12) that

$$\partial\sigma/\partial\alpha = -\tilde{X}/\Lambda, \qquad \partial\sigma/\partial\beta = -\tilde{Y}/\Lambda.$$

On the curve $y = 0$

$$\tilde{X} = 0, \qquad \tilde{Y} = \sin(f(x) - x), \qquad \Lambda = 1 - \cos(f(x) - x)$$

and consequently, by the chain rule,

$$\frac{d\tau}{dx}=\frac{\partial\sigma}{\partial\alpha}\cdot 0+\frac{\partial\sigma}{\partial\beta}\frac{df}{dx}=\frac{-\sin(f(x)-x)}{1-\cos(f(x)-x)}\frac{df}{dx}. \tag{19}$$

The function τ is used in our method for computing σ. Geometrically it determines the curve of intersection of the reflector with the plane $Z=0$. This curve Γ, together with the curves l, L, are shown diagrammatically in figure 2(b).

In order to explain our method for choosing $f(t)$ we have to make some calculations. First of all it is clear that, on the curve l, $\partial\alpha/\partial\phi=0$, $\partial\beta/\partial\phi=f'(t)$. We calculate the other derivatives from equations (2) and (7). Equation (2) shows at once that $\partial\alpha/\partial\theta=D/f'(t)$ on l. We have already seen in this section that on l

$$A=E=0, \qquad B=C=\cos(f(t)-t)-1.$$

It follows from equation (7) that $\partial\beta/\partial\theta=0$. We collect these values together. On the curve l

$$\partial\alpha/\partial\theta=D/f'(t), \qquad \partial\alpha/\partial\phi=\partial\beta/\partial\theta=0, \qquad \partial\beta/\partial\phi=f'(t). \tag{20}$$

The values of the derivatives given in (20) show that, under the transformation χ, tangent vectors to the sphere in the direction of l are stretched by a factor $f'(t)$, whilst those perpendicular to that direction are stretched by a factor $D/f'(t)$. We keep these distortions the same by choosing f so that

$$df/dt=\sqrt{D}.$$

The function f can still be altered by an additive constant. In order to try to avoid source blockage along the initial curves, we arrange that the reflected ray in the direction of the Y axis comes from the incident ray which makes an angle of $-\pi/3$ with the X axis. Thus $f(\pi/2)=-\pi/3$, and f is uniquely determined by this further condition.

As an example of our procedure, we consider the problem of designing a reflector for which

$$D=16\sin^2\theta\sin^2\phi/\cosh^2(6\cos\theta)\cosh^2(6\cos\phi).$$

The initial curve l defined by $\theta=\pi/2$, $\phi=t$ is restricted to the interval $\pi/3\leqslant t\leqslant 2\pi/3$. The function $f(t)$ is determined by integrating the equation

$$\frac{df}{dt}=\frac{4\sin t}{\cosh(6\cos t)}$$

subject to the condition $f(\pi/2)=-\pi/3$. We obtain

$$f(t)=-\tfrac{4}{3}\tan^{-1}[\exp(6\cos t)]$$

so that $f(t)$ lies in the interval $(2\pi/3, 0)$. Consequently $\cos(f(t)-t)\neq 1$ and, as $df/dt\neq 0$, the condition (18) is satisfied. The function τ can be obtained from (19), but the integration has to be done numerically.

6. Reflector curvature

The aim of this section is to calculate the principal curvatures of the reflector along the curve Γ defined in § 5, as these quantities will give us some idea of the shape of the reflector near Γ. However, first of all, we shall obtain some general formulae. We refer to Lipschutz (1969) for the ideas from differential geometry with which we shall be concerned.

We use the notation of § 2 and, in addition, suppose that the reflector is parametrized by coordinates x_1, x_2. For the moment these are general coordinates but they will eventually be specialized to the spherical polar coordinates of the point P. The symbols $\mathrm{d}/\mathrm{d}x^i$, $i = 1, 2$ will denote the partial derivatives of a function whose variables have been expressed in terms of x^1, x^2 only. The principal curvatures are obtained from the first and second fundamental forms of the reflector surface, so we start by finding these latter quantities.

The matrix γ_{ij} of the first fundamental form (with respect to the coordinate system x_1, x_2) is defined by

$$\gamma_{ij} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}x^i} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}x^j}.$$

Using the fact that $\boldsymbol{x} = r\boldsymbol{z}$, we find that

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}x^i} = r\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x^i}\boldsymbol{z} + \sum_k \frac{\mathrm{d}v^k}{\mathrm{d}x^i}\boldsymbol{z}_k\right), \qquad k = 1, 2.$$

The formula (5) shows that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x^i} = \sum_k \Psi_k \frac{\mathrm{d}v^k}{\mathrm{d}x^i}$$

where $\Psi_k = \partial\Psi/\partial v^k$, and consequently

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}x^i} = r\sum_k (\Psi_k \boldsymbol{z} + \boldsymbol{z}_k)\frac{\mathrm{d}v^k}{\mathrm{d}x^i}. \tag{21}$$

It follows that

$$\gamma_{ij} = r^2 \sum_{k,l} (\Psi_k\Psi_l + h_{kl})\frac{\mathrm{d}v^k}{\mathrm{d}x^i}\frac{\mathrm{d}v^l}{\mathrm{d}x^j}, \qquad k, l = 1, 2. \tag{22}$$

The second fundamental form depends on the choice of a unit normal to the reflector surface. The normal $\boldsymbol{n}$ shown in figure 1 is in the direction of $\boldsymbol{y} - \boldsymbol{z}$. Because

$$(\boldsymbol{y}-\boldsymbol{z}) \cdot (\boldsymbol{y}-\boldsymbol{z}) = 2(1 - \boldsymbol{y} \cdot \boldsymbol{z}) = 2\Lambda$$

it follows that

$$\boldsymbol{n} = (\boldsymbol{y}-\boldsymbol{z})/(2\Lambda)^{1/2}. \tag{23}$$

The matrix H_{ij} of the second fundamental form is defined by

$$H_{ij} = -\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}x^i} \cdot \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}x^j}.$$

From equation (23) we find that

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}x^i} = (2\Lambda)^{-1/2}\left(\sum_l \boldsymbol{y}_l \frac{\mathrm{d}u^l}{\mathrm{d}x^i} - \boldsymbol{z}_l \frac{\mathrm{d}v^l}{\mathrm{d}x^i}\right) + \text{a multiple of } \boldsymbol{n}.$$

Therefore, using (21), we have that

$$H_{ij}=r(2\Lambda)^{-1/2}\sum_{k,l}(\Psi_k z+z_k)\,.\left(z_l\frac{\mathrm{d}v^l}{\mathrm{d}x^i}-y_l\frac{\mathrm{d}u^l}{\mathrm{d}x^j}\right)\frac{\mathrm{d}v^k}{\mathrm{d}x^i}.$$

We simplify this expression using

$$\Lambda\Psi_k=\mathbf{y}\,.\,z_k,\qquad \Lambda^2\Psi_{kl}=\Lambda z_k\,.\,y_l+(z_k\,.\,\mathbf{y})(z\,.\,y_l)$$

to obtain

$$H_{ij}=r(2\Lambda)^{-1/2}\sum_{k,l}\left(h_{kl}\frac{\mathrm{d}v^l}{\mathrm{d}x^j}-\Lambda\Psi_{kl}\frac{\mathrm{d}u^l}{\mathrm{d}x^j}\right)\frac{\mathrm{d}v^k}{\mathrm{d}x^i}$$

and then, because

$$\frac{\mathrm{d}u^m}{\mathrm{d}x^j}=\sum_l\frac{\partial u^m}{\partial v^l}\frac{\mathrm{d}v^l}{\mathrm{d}x^j},\qquad m=1,2$$

we can write

$$H_{ij}=r(2\Lambda)^{-1/2}\sum_{k,l}\left(h_{kl}-\sum_m\Lambda\Psi_{km}\frac{\partial u^m}{\partial v^l}\right)\frac{\mathrm{d}v^k}{\mathrm{d}x^i}\frac{\mathrm{d}v^l}{\mathrm{d}x^j}. \tag{24}$$

The formulae (22) and (24) are simplified if we use the coordinates v^1, v^2 of P for the coordinates of the point R on the reflector. For with $x^1=v^1$, $x^2=v^2$ they become

$$\gamma_{ij}=r^2(\Psi_i\Psi_j+h_{ij}) \tag{25}$$

$$H_{ij}=r(2\Lambda)^{-1/2}\left(h_{ij}-\sum_m\Lambda\Psi_{im}\frac{\partial u^m}{\partial v^j}\right). \tag{26}$$

We now calculate the matrices (25) and (26) along the curve Γ defined in § 5, using the spherical polar coordinate system introduced in § 3. In the notation of § 5, the curve Γ is parametrized by t, the spherical polar coordinates of a point on it being given by $(\rho(t), \pi/2, f(t))$, where $\rho(t)=\exp\tau(t)$.

It is convenient to introduce the angle $\delta=f(t)-t$. Then, working from the formulae at the beginning of § 3, we find that corresponding to the above point on Γ

$$h_{11}=h_{22}=1,\qquad h_{12}=h_{21}=0,\qquad \Lambda=1-\cos\delta$$

$$\Psi_1=0,\qquad \Psi_2=-\sin\delta/(1-\cos\delta)=-\cot\tfrac{1}{2}\delta$$

$$\Lambda\Psi_{12}=\Lambda\Psi_{21}=0,\qquad \Lambda\Psi_{11}=-\Lambda\Psi_{22}=1.$$

We shall need to know the values of the partial derivatives $\partial u^i/\partial v^j$ corresponding to a point of Γ. In our case the matrix

$$\left[\frac{\partial u^i}{\partial v^j}\right]=\begin{bmatrix}\partial\theta/\partial\alpha & \partial\theta/\partial\beta\\ \partial\phi/\partial\alpha & \partial\phi/\partial\beta\end{bmatrix}=\begin{bmatrix}\partial\alpha/\partial\theta & \partial\alpha/\partial\phi\\ \partial\beta/\partial\theta & \partial\beta/\partial\phi\end{bmatrix}^{-1}.$$

The appropriate values of the derivatives in the last matrix have been listed in (20) and it follows that along Γ

$$\left[\frac{\partial u^i}{\partial v^j}\right]=\begin{bmatrix}f'/D & 0\\ 0 & 1/f'\end{bmatrix}.$$

We substitute for all these quantities in (25) and (26) and obtain

$$\gamma_{ij} = \rho^2 \begin{bmatrix} 1 & 0 \\ 0 & \operatorname{cosec}^2 \frac{1}{2}\delta \end{bmatrix}$$

$$H_{ij} = \frac{\rho}{2|\sin \frac{1}{2}\delta|} \begin{bmatrix} 1 - f'/D & 0 \\ 0 & 1 + 1/f' \end{bmatrix}.$$

The principal curvatures of the reflector surface are given by the eigenvalues of the matrix H_{ij} with respect to the matrix γ_{ij}. Consequently the principal curvatures at the points on Γ are

$$\frac{1 - f'/D}{2\rho|\sin \frac{1}{2}\delta|}, \qquad \frac{(1 + 1/f')|\sin \frac{1}{2}\delta|}{2\rho}.$$

The principal curvatures at a point on a surface are, by definition, the extreme values of the curvatures of the normal sections of the surface at the point. Suppose that the choice $f' = \sqrt{D}$ has been made. Then, at points on Γ for which $D > 1$, both principal curvatures are positive, so all normal sections bend in the direction of the unit normal $\boldsymbol{n}$. At points on Γ for which $D < 1$, the horizontal section (by the plane $Z = 0$) bends in the direction of $\boldsymbol{n}$, but the vertical section bends away from $\boldsymbol{n}$. Consequently the surface at such points is shaped like a saddle.

7. Conclusions

The design of a reflector surface capable of producing a generalized two-variable beam shape when illuminated by a point source is shown, under the geometric-optics approximation, to require the solution of certain non-linear partial differentiation equations. For a given far-field power density pattern the equations can be of elliptic or hyperbolic type.

The present paper develops a method for the theoretical and numerical solution of these equations in the hyperbolic case. A solution is obtained subject to initial data, and a reflector surface can be designed by suitably limiting the solution. Although no practical designs are presented in the paper we have tested the method on a number of examples and hope to publish the results later.

Features of the method are that the existence and uniqueness of solutions are deduced from classical theorems and the numerical solution is achieved by standard techniques.

There appear to be two main difficulties. In the first place, it is not clear how to choose the initial data in the best way. We have made some comments on this point in § 5. The work in § 6 which deals with the influence of the initial data on the curvature of the reflector surface is also relevant. Secondly, having chosen the initial data, it is not clear *a priori* that the solution will extend far enough from the initial data to allow for the design of a reasonable reflector.

The theory presented here complements the previous work of Westcott and Norris (1975) in the elliptic case. It should be applicable to a wide variety of problems in optics and microwaves in which a shaped beam is required. It should be noted in particular that the theory does not exclude an anisotropic point source as tapered illumination functions for I can be used within the definition of D.

Appendix

Lemma $|\Psi_{ij}| = -(gh)^{1/2}/\Lambda^2$

Proof We shall use the notation of § 2. We write

$$z = by + \sum_j b_j y_j \tag{A.1}$$

$$z_i = a_i y + \sum_j a_{ij} y_j \tag{A.2}$$

and obtain at once by scalar multiplication by y, y_k

$$z \cdot y = b, \qquad z \cdot y_k = \sum_j b_j g_{jk}$$

$$z_i \cdot y = a_i, \qquad z_i \cdot y_k = \sum_j a_{ij} g_{jk}.$$

If $\bar{g}_{ij}$ denotes the matrix inverse to g_{ij}, we find

$$\sum_k z \cdot y_k \bar{g}_{ki} = b_i, \qquad \sum_k z_i \cdot y_k \bar{g}_{kj} = a_{ij}.$$

Consequently because

$$\Psi_{ij} = \frac{\partial^2}{\partial v^i \, \partial u^j}(-\log \Lambda) = \frac{z_i \cdot y_j}{\Lambda} + \frac{(z_i \cdot y)(y_j \cdot z)}{\Lambda^2}$$

it follows that

$$\sum_j \Lambda \Psi_{ij} \bar{g}_{jk} = a_{ik} + a_i b_k / \Lambda. \tag{A.3}$$

The triple scalar products $[z, z_1, z_2]$, $[y, y_1, y_2]$ are related by

$$[z, z_1, z_2] = \begin{vmatrix} b & b_1 & b_2 \\ a_1 & a_{11} & a_{12} \\ a_2 & a_{21} & a_{22} \end{vmatrix} [y, y_1, y_2]$$

so that the determinant

$$\begin{vmatrix} b & b_1 & b_2 \\ a_1 & a_{11} & a_{12} \\ a_2 & a_{21} & a_{22} \end{vmatrix} = \left(\frac{h}{g}\right)^{1/2} \tag{A.4}$$

The lemma follows from (3) and (4), when we have shown that

$$|a_{ik} + a_i b_k / \Lambda| = - \begin{vmatrix} b & b_1 & b_2 \\ a_1 & a_{11} & a_{12} \\ a_2 & a_{21} & a_{22} \end{vmatrix}. \tag{A.5}$$

To accomplish this, we first note that the determinant on the left of (5) is equal to the determinant

$$\begin{vmatrix} 1 & -b_1/\Lambda & -b_2/\Lambda \\ a_1 & a_{11} & a_{12} \\ a_2 & a_{21} & a_{22} \end{vmatrix}. \tag{A.6}$$

This fact is evident when we subtract a_1 times the first row from the second row, and a_2 times the first row from the third row. Now, returning to (1) and (2) we use the facts that z is a unit vector and is orthogonal to z_i to obtain

$$1 = b^2 + \sum_{i,j} g_{ij} b_i b_j \tag{A.7}$$

$$0 = a_i b + \sum_{j,k} a_{ij} g_{jk} b_k. \tag{A.8}$$

We write $l_j = \Sigma_k \, g_{jk} b_k$ and carry out the following operations on the determinant (6). Multiply the second column by l_1, the third column by l_2 and add to the first column. We obtain, using (7) and (8),

$$\begin{vmatrix} 1+(b^2-1)/\Lambda & -b_1/\Lambda & -b_2/\Lambda \\ a_1(1-b) & a_{11} & a_{12} \\ a_2(1-b) & a_{21} & a_{22} \end{vmatrix}.$$

Since $\Lambda = 1 - b$, this determinant is

$$\begin{vmatrix} -b & -b_1/\Lambda & -b_2/\Lambda \\ a_1\Lambda & a_{11} & a_{12} \\ a_2\Lambda & a_{21} & a_{22} \end{vmatrix}$$

which is obviously equal to

$$-\begin{vmatrix} b & b_1 & b_2 \\ a_1 & a_{11} & a_{12} \\ a_2 & a_{21} & a_{22} \end{vmatrix}.$$

We have therefore established (5) and the lemma is proved.

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